# A random integral calculus on generalized s-selfdecomposable probability measures.

Zbigniew J. Jurek

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**Abstract.** It is known that the class $\mathcal{U}_\beta$, of generalized s-selfdecomposable probability distributions, can be viewed as an image via random integral mapping $\mathcal{J}^\beta$ of the class $ID$ of all infinitely divisible measures. We prove that a composition of the mappings $\mathcal{J}^{\beta_1}, \mathcal{J}^{\beta_2}, ..., \mathcal{J}^{\beta_n}$ is again random integral mapping but with a new inner time. In a proof some form of Lagrange interpolation formula is needed. Moreover, some elementary formulas concerning the distributions of products of powers of independent uniformly distributed random variables as established as well.

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*Abbreviated title:* On generalized s-selfdecomposability

Let a real $\beta$ and infinitely divisible probability measures $\nu_j$ (on Euclidean space $\mathbb{R}^d$ or a Banach space $E$) be such that

$$T_{1/n}(\nu_1 * \nu_2 * ... * \nu_n)^{*n^{-\beta}} * \delta_{x_n} \Rightarrow \mu\,, \quad \text{as} \quad n \to \infty\,. \qquad (*)$$

Then we write $\mu \in \mathcal{U}_\beta$ and call $\mu$ *a generalized s-selfdecomposable measure*. In a series of papers Jurek (1988, 1989), Jurek and Schreiber (1992) it was proved, among others, that for not degenerate $\mu$ in $(*)$ we must have $\beta \geq -2$ and that $\mathcal{U}_\beta$ $(\beta \geq -2)$ form an increasing family of convolution semigroups that "almost" exhaust the whole class $ID$ of all infinitely divisible measures.

However, for the purpose of this paper the most crucial is the fact that generalized s-selfdecomposable measures admit the *random integral representation* (1), (see below), which is a particular case of the following representations

$$\mu \equiv I^{h,r}_{(a,b]}(\nu) := \mathcal{L}\big(\int_I h(t)dY_\nu(r(t))\big), \qquad (**)$$

*where* $I = (a,b] \subset \mathbb{R}^+$, $h : \mathbb{R}^+ \to \mathbb{R}$, $Y_\nu(\cdot)$ *is a Lévy process and,*
$r : \mathbb{R}^+ \to \mathbb{R}^+$*is a monotone function (deterministic time change in* $Y_\nu$*)* .

In fact, it was proved that many classes of limit laws can be described as collections of probability distributions of random integrals of the form $(**)$. Later on, this lead to the conjecture that *all classes of limit laws* should admit random integral representation; cf. Jurek (1985; 1988) and see the Conjecture on www.math.uni.wroc.pl/∼zjjurek [1]. More recently the method of random integral representation was used among others by Aoyama and Maejima (2006), Maejima and Sato (2009). Here we give yet another example of calculus on some infinitely divisible laws.

In this paper we will prove that the class of the integral mappings (1), for $\beta > 0$, is closed under compositions, that is, their compositions are of the form $(**)$ with the properly chosen time change $r$; (Theorem 1, Proposition 2). As an auxiliary result we found a decomposition of number 1 as a sum of products of complex fractions; (Proposition 1). Also compositions of the mappings (1) are described in terms of the Lévy-Khintchine triples: (Theorem 2). Auxiliary Lemmas 2 and 3 give probability distribution functions (p.d.f.) of products of powers of independent uniformly distributed random variables as linear combinations of other p.d.f.

## 1. Introduction and main results.

*The results here are given for random vectors in* $\mathbb{R}^d$ *. However, proofs are such that they are valid for infinite dimensional separable Banach spaces* $E$ *when ones replaces a scalar product by the bilinear form on the product space* $E^{'} \times E$*, where* $E^{'}$ *denotes the dual space. Of course,* $(\mathbb{R}^d)^{'} = \mathbb{R}^d$.

Throughout the paper $\mathcal{L}(X)$ will denote the probability distribution of an $\mathbb{R}^d$-valued random vector $X$; (or a real separable Banach space E-valued

[1] It might be of an interest to recall here that S. D. Chatterji's subsequence principle claiming that: *Given a limit theorem for independent identically distributed random variables under certain moment conditions, there exists an analogous theorem such that an arbitrary-dependent sequence (under the same moment conditions) always contains a subsequence satisfying this analogous theorem* was proved by David J. Aldous (1977). Although, we do not expect that the above Conjecture and Chatterji's subsequence principle are mathematically related, however, one may see a "philosophical" relation between those two.

random element $X$ if the Reader is interested in that generality). Similarly, by $Y_\nu(t), t \geq 0$, we will denote an $\mathbb{R}^d$- valued (or an E-valued) Lévy stochastic process such that $\mathcal{L}(Y_\nu(1)) = \nu$. Recall that by a Lévy stochastic process we mean a process with stationary independent increments, starting from zero, and with paths that are continuous from the right and with left limits (that is, cadlag paths). Of course, $\nu \in ID$, where $ID$ stands for all *infinitely divisible* measures on $\mathbb{R}^d$ (or on a Banach space $E$).

For $\beta > 0$ and a Lévy process $Y_\nu(t), t \geq 0$, we define mappings

$$\mathcal{J}^{\{\beta\}}(\nu) \equiv \mathcal{J}^{\beta}(\nu) := \mathcal{L}\big(\int_{(0,1]} t^{1/\beta}\, dY_\nu(t)\big) = \mathcal{L}\big(\int_{(0,1]} t\, dY_\nu(t^{\beta})\big) \qquad (1)$$

and the classes $\mathcal{U}_\beta := \mathcal{J}^{\beta}(ID)$. To the distributions from $\mathcal{U}_\beta$ we refer to as *generalized s-selfdecomposable distributions.*

These classes of probability measures were originally defined as limiting distributions in some schemes of summations; cf. Jurek (1988 and 1989). In particular, the class $\mathcal{U} \equiv \mathcal{U}_1$ of s-selfdecomposable was defined by non-linear shrinking operations $U_r, r > 0$, (for $x > 0$, $U_r(x) := \max(0, x - r)$); cf. Jurek (1981).

**Remark 1.** Since the process $Y$ has values in a metric separable complete space we may and do assume that the paths of $Y$ are cadlag; cf. Theorem A.1.1 in Jurek and Vervaat (1983), p.260. Since the random integral in (1) is defined by a formal integration by parts formula, therefore the random integral in question do exist; cf. Jurek-Vervaat (1983), Lemma 1.1 or Jurek-Mason (1993), Lemma 3.6.4, on p. 119.

For a positive natural $m$ and a sequence of positive real $\beta_1, \beta_2, ..., \beta_m$ and a probability measure $\nu \in ID$, let us define the mappings

$$\mathcal{J}^{\{\beta_1,\ldots,\beta_m\}}(\nu) := \mathcal{J}^{\beta_m}(\mathcal{J}^{\{\beta_1,\ldots,\beta_{m-1}\}}(\nu)) \; = \mathcal{L}\big(\int_{(0,1]} t\, dY_{\mathcal{J}^{\{\beta_1,\ldots,\beta_{n-1}\}}(\nu)}(t^{\beta_m})\big)\, .$$

Our main results say that the above compositions can be written as an integral of the form $(**)$ with a suitable chosen time change $r$. Furthermore, compositions are expressed in terms of the individual random integrals.

**Theorem 1.** *For positive integers $\beta_1, \beta_2, ..., \beta_m$ and an infinitely divisible probability measure $\nu$ we have*

$$\mathcal{J}^{\{\beta_1,\ldots,\beta_n\}}(\nu) \; = \mathcal{L}\big(\int_{(0,1]} t\, dY_\nu(r_{\{\beta_1,\ldots,\beta_m\}}(t)\big) = I_{(0,1]}^{t,\, r_{\{\beta_1,\ldots,\beta_m\}}}(\nu) \qquad (2)$$

*and the time scale change* $r_{\{\beta_1,\ldots,\beta_m\}}$ *is given by*

$$r_{\{\beta_1,\ldots,\beta_m\}}(t) = \mathbb{P}\,[\,U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot\ldots\cdot U_m^{1/\beta_m} \le t\,],\quad 0<t\le 1\,, \tag{3}$$

*where* $U_i$*'s are stochastically independent random variables uniformly distributed over the unit interval.*

*If all* $\beta_1,\ldots,\beta_n$ *are different then*

$$r_{\{\beta_1,\ldots,\beta_n\}}(t) := \sum_{j=1}^{n} C_{j,n}\, t^{\beta_j},\quad C_{j,n}(\beta_1,\ldots,\beta_n)\equiv C_{j,n} := \prod_{k\neq j,k=1}^{n} \frac{\beta_k}{\beta_k-\beta_j} \tag{4}$$

*and, in particular, we get the equality:* $\sum_{j=1}^{n} C_{j,n} = 1$ .

*If* $\beta_1=\beta_2=\ldots=\beta_m=\alpha$ $(m\ge 1)$ *then*

$$r_{\{\underbrace{\alpha,\ldots,\alpha}_{m-times}\}}(t) = t^{\alpha}\sum_{j=0}^{m-1}\frac{(-\alpha\log t)^j}{j!},\quad \textit{for } 0<t\le 1\,, \tag{5}$$

(Other collections of $\beta$'s are discussed below.)

**Remark 2.** Note that if $\mathcal{E}(\lambda)$ denotes the exponential random variable with the parameter $\lambda$ then $e^{-\mathcal{E}(\lambda)} \stackrel{d}{=} U^{1/\lambda}$. Hence if $\mathcal{E}_i(\alpha), 1\le i\le m$, independent and identically distributed exponential random variables then $r_{\{\alpha,\ldots,\alpha\}}$ is the cumulative distribution of $e^{-\mathcal{E}_1(\alpha)}\cdot e^{-\mathcal{E}_2(\alpha)}\cdot\ldots\cdot e^{-\mathcal{E}_m(\alpha)} \stackrel{d}{=} e^{-\gamma_{m,\alpha}}$, where $\gamma_{m,\alpha}$ is the gama random variable with the shape parameter $m$ and the scale parameter $\alpha$, i.e., it has the probability density function of the form $\alpha^m/(m-1)!\; x^{m-1}\,e^{-\alpha x}\,1_{(0,\infty)}(x)$; in particular, for $\alpha=1$ comp. Proposition 4 in Jurek (2004).

In a proof of the above theorem the following identity, that might be also of an independent interest, is needed.

**Proposition 1.** *For different complex numbers* $z_j, j=1,2,\ldots,n,n+1$ *we have equality:*

$$\sum_{i=1}^{n}\frac{1}{z_i-z_{n+1}}\Big(\prod_{k=1;k\neq i}^{n}\frac{1}{z_k-z_i}\Big) = \prod_{i=1}^{n}\frac{1}{z_i-z_{n+1}}. \tag{6}$$

*Equivalently, for any different complex numbers* $z_j, j=1,2,\ldots,n,$ *we have the identity*

$$\sum_{i=1}^{n}\prod_{k=1;k\neq i}^{n}\frac{z_k-z}{z_k-z_i}\equiv 1,\quad \textit{for all}\quad z\in\mathbb{C}\,, \tag{7}$$

*that can be regarded as a canonical decomposition of 1 as a sum of finite products of complex fractions.*

Since the characteristic function of each $\nu \in ID$ is uniquely determined by the triple $[a, R, M]$ from its Lévy-Khintchine formula we will write formally that $\nu = [a, R, M]$; for details see the Section 2.1 below.

If $\nu = [a, R, M]$ and $\mathcal{J}^{\{\beta\}}(\nu) := [a^{\{\beta\}}, R^{\{\beta\}}, M^{\{\beta\}}]$ and

$$b_{M,\beta} := \int_{\{||x||>1\}} x\, ||x||^{-1-\beta}\, M(dx) \in \mathbb{R}^d \text{ ( or a Banach space } E) \qquad (8)$$

then we have

$$a^{\{\beta\}} = \beta(\beta+1)^{-1}(a + b_{M,\beta}), \quad R^{\{\beta\}} = \beta(2+\beta)^{-1}R,$$
$$M^{\{\beta\}}(A) = \int_0^1 T_{t^{1/\beta}}M(A)dt = \beta \int_0^1 M(s^{-1}A)\, s^{\beta-1}ds, \text{ for } A \in \mathcal{B}_0, \quad (9)$$

where $\mathcal{B}_0$ stands for all Borel subsets of $\mathbb{R}^d \setminus \{0\}$ (or $E \setminus \{0\}$.

With these notations Theorem 1 gives the description of random integrals (2) in terms of the corresponding triples.

**Theorem 2.** *For distinct positive reals $\beta_1, \beta_2, ..., \beta_n$, coefficients $C_{j,n}$ defined by (4), an infinitely divisible probability measure $\nu = [a, R, M]$ and $\mathcal{J}^{\{\beta_1,...,\beta_n\}}(\nu) = [a^{\{\beta_1,...,\beta_n\}}, R^{\{\beta_1,...,\beta_n\}}, M^{\{\beta_1,...,\beta_n\}}]$ we have*

$$a^{\{\beta_1,...,\beta_n\}} = a\prod_{j=1}^{n} \frac{\beta_j}{\beta_j+1} + \sum_{j=1}^{n} \frac{\beta_j\, b_{M,\beta_j}}{\beta_j+1} \prod_{k\neq j, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_j} = \sum_{j=1}^{n} C_{j,n}\, a^{\{\beta_j\}} \quad (10)$$

$$R^{\{\beta_1,...,\beta_n\}} = \prod_{j=1}^{n} \frac{\beta_j}{\beta_j+2}\, R = \sum_{j=1}^{n} C_{j,n} R^{\{\beta_j\}}, \qquad (11)$$

$$M^{\{\beta_1,...,\beta_n\}}(A) = \int_0^1 ... \int_0^1 T_{t_1^{1/\beta_1} ... t_n^{1/\beta_n}} M(A)dt_1 ... dt_n = \sum_{j=1}^{n} C_{j,n}\, M^{\{\beta_j\}}(A) \qquad (12)$$

*where $b_{M,\beta_j}$, $a^{\{\beta_j\}}$, $R^{\{\beta_j\}}$ and $M^{\{\beta\}}$ are given in (8) and (9).*

**Corollary 1.** *For different positive reals $\beta_1, \beta_2, ..., \beta_n$ and the constants $C_{j,n}$ given in (4) we have*

$$\mathcal{J}^{\{\beta_1,...,\beta_n\}}(\nu) = (\mathcal{J}^{\{\beta_1\}}(\nu))^{*C_{1,n}} * ... * (\mathcal{J}^{\{\beta_n\}}(\nu))^{*C_{n,n}},$$

*where for $C_{j,n} < 0$ the corresponding convolution power means the reciprocal of the corresponding infinitely divisible Fourier transform.*

Here we have analogous formulae to that of Theorem 1, for the composition of integral mappings $\mathcal{J}^{\beta}$ with different collections of $\beta$'s.

**Proposition 2.** *In the formula (2) in Theorem 1 we have:*
*(a) For distinct positive reals $\beta_1, ..., \beta_n, \alpha$ and integer $m \geq 1$,*

$$r_{\{\beta_1,...,\beta_n,\underbrace{\alpha,...,\alpha}_{m-times}\}}(t) = \sum_{j=1}^{n} d_{j,n}^{(m)}\, r_{\{\beta_j\}}(t)$$
$$- \alpha^{-1} \sum_{k=0}^{m-1} \big( \sum_{j=1}^{n} \beta_j\, d_{j,n}^{(m-k)} \big) r_{\{\underbrace{\alpha,...,\alpha}_{k+1-times}\}}(t), \qquad 0 < t \leq 1, \tag{13}$$

*where the coefficients $d_{j,n}^{(l)}$ are given by*

$$d_{j,n}^{(l)} \equiv d_{j,n}^{(l)}(\beta_1, ..., \beta_n\,; \underbrace{\alpha, ..., \alpha}_{l-times}) = \prod_{k \neq j\,, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_j}\, (\frac{\alpha}{\alpha - \beta_j})^l$$
$$= C_{j,n}(\beta_1, ..., \beta_n)\, (\frac{\alpha}{\alpha - \beta_j})^l, \qquad l = 0, 1, 2, .... \tag{14}$$

*(b) For different positive reals $\alpha$ and $\gamma$, and positive integers $k, l \geq 1$,*

$$r_{\{\underbrace{\alpha,...,\alpha}_{k-times},\underbrace{\gamma,...,\gamma}_{l-times}\}}(t) = (\frac{\alpha}{\alpha-\gamma})^k\, (\frac{\gamma}{\gamma-\alpha})^l\, \Big[ \sum_{s=1}^{k} \frac{(l)_{k-s}}{(k-s)!} (\frac{\alpha-\gamma}{\alpha})^s\, r_{\{\underbrace{\alpha,...,\alpha}_{s-times}\}}(t)$$
$$- \sum_{r=1}^{k+l-1} e_{r,\,k+l} (\frac{\gamma-\alpha}{\gamma})^r\, r_{\{\underbrace{\gamma,...,\gamma}_{r-times}\}}(t) \Big], \qquad for\ \ 0 < t \leq 1, \tag{15}$$

*where to coefficients $e_{r,k+l}$ are given by*

$$e_{r,\,k+l} = \sum_{s=1}^{r \wedge k} (-1)^s \frac{(s)_{r-s}}{(r-s)!} \frac{(l)_{k-s}}{(k-s)!} \qquad for \qquad 1 \leq r \leq k+l\,,$$

*and for $w \in \mathbb{R}$ and $m = 1, 2, ...$, $(w)_m := w(w+1)...(w+m-1)$ denotes the Pochhammer symbol.*

*In particular, we get an identity*

$$(\frac{\alpha}{\alpha-\gamma})^k\, (\frac{\gamma}{\gamma-\alpha})^l\, \Big[ \sum_{s=1}^{k} \frac{(l)_{k-s}}{(k-s)!} (\frac{\alpha-\gamma}{\alpha})^s - \sum_{r=1}^{k+l-1} e_{r,\,k+l} (\frac{\gamma-\alpha}{\gamma})^r \Big] = 1 \tag{16}$$

**Remark 3.** (i) Using the above formulae, and repeating elementary calculations from proofs of Lemmas 2 and 3 one can get $r_{\mathbb{A}}$ for the most general collections $\mathbb{A} = \{\beta_1, \beta_2, ...\beta_n; \underbrace{\alpha_1, ..., \alpha_1}_{m_1-times}, ..., \underbrace{\alpha_r, ..\alpha_r}_{m_r-times}\}$, where $\beta_1, ..., \beta_n, \alpha_1, ..., \alpha_r$

are distinct positive reals. However, expressions are getting quite involved and are not presented here; cf. Section 2.7.

(ii) For a finite set $\mathbb{B}$ of not necessary distinct complex numbers, let us introduce the function $\rho_{\mathbb{B}}$ given by

$$\rho_{\mathbb{B}}(b) := \prod_{c\neq b,\ c\in\mathbb{B}} \frac{c}{c-b}, \qquad b\in\mathbb{B}. \tag{17}$$

Then for the set $\mathbb{A}$ from (i) we have

$$\rho_{\mathbb{A}}(\beta_j) = \prod_{k\neq j,k=1}^{n} \frac{\beta_k}{\beta_k-\beta_j} \prod_{l=1}^{r} (\frac{\alpha_l}{\alpha_l-\beta_j})^{m_l} \ \text{ for } \ 1\le j\le n \tag{18}$$

$$\rho_{\mathbb{A}}(\alpha_s) = \prod_{k=1}^{n} \frac{\beta_k}{\beta_k-\alpha_s} \prod_{l\neq s,l=1}^{r} (\frac{\alpha_l}{\alpha_l-\alpha_s})^{m_l} \ \text{ for } \ 1\le s\le r. \tag{19}$$

Furthermore, if we put $K := n+m_1+m_2+...+m_r$ and define

$$\beta'_j := \begin{cases} \beta_j & \text{for } 1\le j\le n \\ \alpha_1 & \text{for } n+1\le j\le n+m_1 \\ \alpha_2 & \text{for } n+m_1+1\le j\le n+m_1+m_2 \\ ... & \\ \alpha_r & \text{for } K-m_r+1\le j\le K \end{cases}$$

then $\mathbb{A} = \{\beta'_1, \beta'_2, ..., \beta'_K\}$ and hence we get that

$$\rho_{\mathbb{A}}(\beta'_j) = \prod_{\beta'_k\neq\beta'_j,\ k=1}^{K} \frac{\beta'_k}{\beta'_k-\beta'_j} \qquad \text{for } \ 1\le j\le K, \tag{20}$$

which coincides with (18) and (19). The notation of $\rho_{\mathbb{B}}$ may give a more concise way of getting formulae for the general case of $\mathbb{A}$; comp. above note (i). From the property of the constants $C_{j,n}$ in Theorem 1 we conclude that

$$\sum_{b\in\mathbb{B}} \rho_{\mathbb{B}}(b) = 1 \tag{21}$$

whenever $\mathbb{B}$ is a finite set of distinct numbers.

## 2. Auxiliary results and proofs.

**2.1.** Random integrals.

Let us recall that for a probability Borel measures $\mu$ on $\mathbb{R}^d$ (or on $E$), its *characteristic function* ( Fourier transform) $\hat{\mu}$ is defined as

$$\hat{\mu}(y) := \int_{\mathbb{R}^d} e^{i<y,x>}\mu(dx), \ y\in\mathbb{R}^d, \quad (\text{or } \ y\in E')$$

where $< \cdot, \cdot >$ denotes the scalar product; (in case of Banach spaces, $< \cdot, \cdot >$ is the bilinear form on $E' \times E$). Further, the characteristic function of an infinitely divisible probability measure $\mu$ admits the following Lévy-Khintchine representation

$$\hat{\mu}(y) = e^{\Phi(y)},\ y \in \mathbb{R}^d, \ \text{ and the Lévy exponent } \ \Phi(y) = i < y, a > - \\ \frac{1}{2} < y, Ry > + \int_{\mathbb{R}^d \setminus \{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M(dx), \tag{22}$$

where $a$ is a *shift vector*, $R$ is a *covariance operator* corresponding to the Gaussian part of $\mu$ and $M$ is a *Lévy spectral measure*. Since there is a one-to-one correspondence between measures $\mu \in ID$ and triples $a$, $R$ and $M$ in its Lévy-Khintchine formula (22) we will formally write $\mu = [a, R, M]$; cf. Araujo-Giné (1980) or Parthasarathy (1967).

Note that for $s \in \mathbb{R}$ we have

$$\Phi(sy) = i < y, s(a + \int_{E\setminus\{0\}} x(1_B(sx) - 1_B(x))M(dx)) > -\frac{1}{2}s^2 < y, Ry > \\ + \int_{E\setminus\{0\}} [e^{i<y,z>} - 1 - i < y, z > 1_B(z)] M(s^{-1}dz) \tag{23}$$

Finally, let us recall that

$$M \text{ is Lévy spectral measure on } \mathbb{R}^d \text{ iff } \int_{\mathbb{R}^d} \min(1, ||x||^2) M(dx) < \infty \tag{24}$$

For infinity divisibility on Banach spaces we refer to the monograph by Araujo-Giné (1980), Chapter 3, Section 6, p. 136. Let us emphasize here that the characterization (24) of Lévy spectral measures is NOT true on infinite dimensional Banach spaces ! However, it holds true on Hilbert spaces; cf. Parthasarathy (1967), Chapter VI.

For this note it is important to have the following technical result:

**Lemma 1.** *If the random integral $A \equiv \int_{(a,b]} h(t) dY_\nu(r(t))$ exists then we have*

$$\log \widehat{\mathcal{L}(A)}(y) = \int_{(a,b]} \log \widehat{\mathcal{L}(Y_\nu(1))}(h(s)y) dr(s) = \int_{(a,b]} \Phi(h(s)y) dr(s),$$

*where $y \in \mathbb{R}^d$ (or $E'$) and $\Phi$ is the Lévy exponent of $\widehat{\mathcal{L}(Y_\nu(1))} = \hat{\nu}$. In particular, if $r$ is the cumulative probability distribution function of a random variable $T$ concentrated of the interval $(a, b]$ then $\log \widehat{\mathcal{L}(A)}(y) = \mathbb{E}[\Phi(h(T)y)]$.*

The formula in Lemma 1 is a straightforward consequence of our definition (integration by parts) of the random integrals (∗∗). The proof is analogous to that in Jurek-Vervaat(1983), Lemma 1.1 or Jurek-Mason (1993), Lemma 3.6.4 or Jurek (1988), Lemma 2.2 (b).

**Remark 4.** Note that for bounded intervals $(a, b] \subset \mathbb{R}^+$, continuous $h$ and $r$ of bounded variation, the integrals of the form $A$ in Lemma 1 are well-defined.

**2.2.** Proof of Proposition 1.

Firstly, assuming the convention that $\prod_{i\in\emptyset} z_i = 1$ then the statement (6) is true for $n = 1$.

Secondly, note that the last value $z_{n+1}$ is not 'reached' by any other $z_i$, because $1 \le i \le n$. It might be treated as a number independent of the index $i$. Hence, by an induction assumption, the equality (6) for different complex numbers $z_1, z_2, .., z_{n-1}$ and the last values equal to $z_n$ and $z_{n+1}$, respectively, gives that

$$\sum_{i=1}^{n-1} \frac{1}{z_i - z_n}\Big(\prod_{k=1;k\neq i}^{n-1} \frac{1}{z_k - z_i}\Big) = \prod_{k=1}^{n-1} \frac{1}{z_k - z_n} \tag{25}$$

and

$$\sum_{i=1}^{n-1} \frac{1}{z_i - z_{n+1}}\Big(\prod_{k=1;k\neq i}^{n-1} \frac{1}{z_k - z_i}\Big) = \prod_{k=1}^{n-1} \frac{1}{z_k - z_{n+1}} \tag{26}$$

Consequently, for different $n + 1$ complex numbers $z_1, z_2, ..., z_n, z_{n+1}$ we get

$$\sum_{i=1}^{n} \tfrac{1}{z_i - z_{n+1}}\Big(\prod_{k=1;k\neq i}^{n} \tfrac{1}{z_k - z_i}\Big) =$$

$$= \sum_{i=1}^{n-1} \tfrac{1}{z_i - z_{n+1}}\Big(\prod_{k=1;k\neq i}^{n-1} \tfrac{1}{z_k - z_i} \cdot \tfrac{1}{z_n - z_i}\Big) + \tfrac{1}{z_n - z_{n+1}} \prod_{k=1;k\neq n}^{n} \tfrac{1}{z_k - z_n}$$

$$= \sum_{i=1}^{n-1} \tfrac{1}{z_i - z_{n+1}} \cdot \tfrac{1}{z_n - z_i}\Big(\prod_{k=1;k\neq i}^{n-1} \tfrac{1}{z_k - z_i}\Big) + \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \tfrac{1}{z_k - z_n}$$

$$= \sum_{i=1}^{n-1}\Big(\tfrac{1}{z_i - z_{n+1}} + \tfrac{1}{z_n - z_i}\Big)\tfrac{1}{z_n - z_{n+1}}\Big(\prod_{k=1;k\neq i}^{n-1} \tfrac{1}{z_k - z_i}\Big) + \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \tfrac{1}{z_k - z_n}$$

$$= \tfrac{1}{z_n - z_{n+1}} \sum_{i=1}^{n-1} \tfrac{1}{z_i - z_{n+1}} \Big( \prod_{k=1;k\neq i}^{n-1} \tfrac{1}{z_k - z_i} \Big) + \tfrac{1}{z_n - z_{n+1}} \sum_{i=1}^{n-1} \tfrac{1}{z_n - z_i} \Big( \prod_{k=1;k\neq i}^{n-1} \tfrac{1}{z_k - z_i} \Big)$$

$$+ \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \tfrac{1}{z_k - z_n} \qquad \text{by (25) and (26)}$$

$$= \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \frac{1}{z_k - z_{n+1}} - \tfrac{1}{z_n - z_{n+1}} \sum_{i=1}^{n-1} \tfrac{1}{z_i - z_n} ( \prod_{k=1;k\neq i}^{n-1} \tfrac{1}{z_k - z_i} \Big) + \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \tfrac{1}{z_k - z_n}$$

$$= \prod_{k=1}^{n} \frac{1}{z_k - z_{n+1}} - \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \tfrac{1}{z_k - z_n} + \tfrac{1}{z_n - z_{n+1}} \prod_{k=1}^{n-1} \tfrac{1}{z_k - z_n} = \prod_{k=1}^{n} \frac{1}{z_k - z_{n+1}},$$

which proves the formula (6).

By multiplying both sides of (6) by the product $\prod_{i=1}^{n}(z_i - z_{n+1})$ we arrive at identity (7) for arbitrary $z = z_{n+1}$ different from all $z_j$, $j = 1, 2, ..., n$. For $z = z_l$, for some $1 \leq l \leq n$, in (6) the term $i = l$ is equal 1 and all others are zero. This completes a proof of Proposition 1.

**Remark 5.** (a) Note that it is enough to prove (7) for $z = 0$. This is so, because we may write (7) for a new constants $\tilde{z}_l := z_l + z$.
(b) In the interpolation theory of functions for given set of points $(x_0, y_0)$, $(x_1, y_1), ..., (x_n, y_n)$ in the plane $\mathbb{R}^2$, with distinct $x_0, x_1, ..., x_n$,

$$P_n(x) := \sum_{j=0}^{n} y_j \prod_{k\neq j,k=0}^{n} \frac{x_k - x}{x_k - x_j},$$

is the unique Lagrange interpolating polynomial of degree less or equal $n-1$ and such that

$$P_n(x_i) = y_i \ \text{ for all } \ i = 0, 1, 2, ..., n;$$

cf. Kincaid and Cheney (1996), Chapter 6. Thus for the particular points $(x_0, 1), (x_1, 1), ..., (x_n, 1)$ in $\mathbb{R}^2$ we get the line $y = P_n(x) = 1$ as the Lagrange interpolating polynomial. However, our Proposition 1 is for complex coefficients with completely different proof and seems to be not known in the interpolation theory.

**2.3.** Products of uniform distributions: part I.

Here are some elementary identities concerning the products of powers of independent uniformly distributed random variables. The main objective is to express the cumulative distribution function (c.d.f.) or probability density function (p.d.f.) of such products as a linear combinations (with not necessary positive coefficients) of other c.d.f. (or p.d.f.).

**Lemma 2.** *Let $U_i$, $1 \le i \le n$ be i.i.d. uniformly distributed over the interval $(0,1]$, $\alpha_i > 0$ and let $f_{\{\alpha_1,\alpha_2,\dots,\alpha_n\}}$ and $F_{\{\alpha_1,\alpha_2,\dots,\alpha_n\}}$ denote the probability density and cumulative distribution function of $U_1^{1/\alpha_1} \cdot U_2^{1/\alpha_2} \cdot \dots \cdot U_n^{1/\alpha_n}$, respectively. Then*

$(a)$ $f_{\{\alpha_1,\alpha_2,\dots,\alpha_n\}}(x_n) =$

$$\alpha_n\, x_n^{\alpha_n-1} \int_{x_n}^{1} \alpha_{n-1}\, x_{n-1}^{\alpha_{n-1}-\alpha_n-1} \int_{x_{n-1}}^{1} \alpha_{n-2}\, x_{n-2}^{\alpha_{n-2}-\alpha_{n-1}-1} \int_{x_{n-2}}^{1} \alpha_{n-3}\, x_{n-3}^{\alpha_{n-3}-\alpha_{n-2}-1} \dots$$

$$\dots \int_{x_3}^{1} \alpha_2\, x_2^{\alpha_2-\alpha_3-1} \int_{x_2}^{1} \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1\, dx_2 \dots dx_{n-2}\, dx_{n-1},\ \ 0 < x_n \le 1. \quad (27)$$

*(b) If $\alpha_1 = \alpha_2 = \dots = \alpha_m = \alpha$ then*

$$f_{\{\underbrace{\alpha, \dots, \alpha}_{m-times}\}}(x) = \alpha\, x^{\alpha-1}\, \frac{(-\alpha\, \log x)^{m-1}}{(m-1)!} \quad \textit{for}\ \ 0 < x \le 1, \quad (28)$$

$$F_{\{\underbrace{\alpha, \dots, \alpha}_{m-times}\}}(s) = s^{\alpha} \sum_{j=0}^{m-1} \frac{(-\alpha \log s)^j}{j!} \quad \textit{for}\ \ 0 < s \le 1, \quad (29)$$

*and $F_{\{\underbrace{\alpha, \dots, \alpha}_{m-times}\}}(s) = 1$ for $s \ge 1$ and zero for $s < 0$.*

*(c) If all positive reals $\alpha_i$ $(i = 1, 2, \dots, n)$ are distinct and*

$$C_{j,n}(\alpha_1, \dots, \alpha_n) \equiv C_{j,n} := \prod_{k \ne j\,, k=1}^{n} \frac{\alpha_k}{\alpha_k - \alpha_j} \quad \textit{and}\ \ c_{j,n} := \prod_{k \ne j\,, k=1}^{n} \frac{1}{\alpha_k - \alpha_j} \quad (30)$$

*then*

$$f_{\{\alpha_1,\alpha_2,\dots,\alpha_n\}}(x) = \sum_{j=1}^{n} C_{j,n}\, \alpha_j x^{\alpha_j-1} = \alpha_1 \dots \alpha_n \sum_{j=1}^{n} c_{j,n} x^{\alpha_j-1},\ \ 0 < x \le 1, \quad (31)$$

*and*

$$F_{\{\alpha_1,\alpha_2,\dots,\alpha_n\}}(s) = \sum_{j=1}^{n} C_{j,n}\ s^{\alpha_j},\ \textit{for}\ 0 < s \le 1, \quad \textit{where} \sum_{j=1}^{n} C_{j,n} = 1. \quad (32)$$

*Proof.* For positive and independent rv $X$ and $Z$ with p.d.f. $f_X$ and $f_Z$, respectively we have that $X \cdot Z$ has the p.d.f.

$$f_{X \cdot Z}(z) = \int_0^{\infty} f_X(\frac{z}{x}) \frac{1}{x} f_Z(x)\, dx\,. \quad (33)$$

Since $f_{\{\alpha\}}(x) = \alpha\, x^{\alpha-1}\, 1_{(0,1)}(x)$ is the p.d.f. of $U^{1/\alpha}$ therefore from (33) we get

$$f_{\{\alpha,\beta\}}(z) = f_{U^{1/\alpha}.U^{1/\beta}}(z) = \beta \int_z^1 f_{\{\alpha\}}(\frac{z}{x}) x^{\beta-2} dx = f_{\{\beta\}}(z) \int_z^1 f_{\{\alpha\}}(t)\, t^{-\beta}\, dt. \tag{34}$$

Hence for $\alpha_1$ and $\alpha_2$ we conclude that

$$f_{\{\alpha_1,\alpha_2\}}(x_2) = \alpha_2\, x_2^{\alpha_2-1} \int_{x_2}^1 \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1 \tag{35}$$

which is indeed of the form (27) for $n = 2$.

Assume, by the induction argument, that the formula (27) holds true for $n$. Then using (33) and (34) we obtain

$$\begin{aligned} f_{\{\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_{n+1}\}}(x_{n+1}) &= f_{\alpha_{n+1}}(x_{n+1}) \int_{x_{n+1}}^1 f_{\{\alpha_1,\alpha_2,\ldots,\alpha_n\}}(x_n)\, x_n^{-\alpha_{n+1}} dx_n = \\ \alpha_{n+1} x_{n+1}^{\alpha_{n+1}-1} \int_{x_{n+1}}^1 \Big( \alpha_n\, x_n^{\alpha_n-1} &\int_{x_n}^1 \alpha_{n-1}\, x_{n-1}^{\alpha_{n-1}-\alpha_n-1} \int_{x_{n-1}}^1 \alpha_{n-2}\, x_{n-2}^{\alpha_{n-2}-\alpha_{n-1}-1} \\ \ldots \int_{x_3}^1 \alpha_2\, x_2^{\alpha_2-\alpha_3-1} \int_{x_2}^1 \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, &dx_1\, dx_2 \ldots dx_{n-2}\, dx_{n-1} \Big)\, x_n^{-\alpha_{n+1}} dx_n = \\ \alpha_{n+1} x_{n+1}^{\alpha_{n+1}-1} \int_{x_{n+1}}^1 \alpha_n\, x_n^{\alpha_n-\alpha_{n+1}-1} &\int_{x_n}^1 \alpha_{n-1}\, x_{n-1}^{\alpha_{n-1}-\alpha_n-1} \int_{x_{n-1}}^1 \alpha_{n-2}\, x_{n-2}^{\alpha_{n-2}-\alpha_{n-1}-1} \\ \ldots \int_{x_3}^1 \alpha_2\, x_2^{\alpha_2-\alpha_3-1} &\int_{x_2}^1 \alpha_1\, x_1^{\alpha_1-\alpha_2-1}\, dx_1\, dx_2 \ldots dx_{n-1}\, dx_n, \end{aligned}$$

which is the equality (27) for $n+1$. Thus the proof of the part (a) is complete.

Taking in (27), $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$ and performing the successive integrations, we get the formula (28). (Or simply prove (28) by the induction argument utilizing (33)). Integrating p.d.f. (28) we get c.d.f. (29). (Recall also here Remark 6 and the formula (39) from below.) Thus the part (b) is proved.

In the part(c), formulae (31) and (32) are obvious for n=1. Assume that (31) holds true for n. First, from Proposition 1 formula (6) we infer that for $1 \le j \le n$ we get

$$c_{j,n}(\alpha_{n+1} - \alpha_j)^{-1} = c_{j,n+1}, \quad \sum_{j=1}^{n} (\alpha_j - \alpha_{n+1})^{-1} c_{j,n} = c_{n+1,n+1}\ . \tag{36}$$

Then from (34), (31) and (36) and again (6) from Proposition 1 we get

$$f_{\{\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha_{n+1}\}}(x) = \alpha_{n+1} x^{\alpha_{n+1}-1} \int_x^1 f_{\{\alpha_1,\alpha_2,\ldots,\alpha_n\}}(t)\, t^{-\alpha_{n+1}} dt$$

$$= \alpha_1 \ldots \alpha_{n+1} \sum_{j=1}^{n} c_{j,n}\ x^{\alpha_{n+1}-1} \int_x^1 t^{\alpha_j-\alpha_{n+1}-1} dt$$

$$= \alpha_1 \ldots \alpha_{n+1} \sum_{j=1}^{n} c_{j,n} \frac{1}{\alpha_j - \alpha_{n+1}} (x^{\alpha_{n+1}-1} - x^{\alpha_j-1})$$

$$= \alpha_1 \ldots \alpha_{n+1} \sum_{j=1}^{n} c_{j,n+1} x^{\alpha_j-1} + \alpha_1 \ldots \alpha_{n+1} \big( \sum_{j=1}^{n} c_{j,n} \frac{1}{\alpha_j - \alpha_{n+1}} \big) x^{\alpha_{n+1}-1}$$

$$= \alpha_1 \ldots \alpha_{n+1} (\sum_{j=1}^{n} c_{j,n+1} x^{\alpha_j-1} + c_{n+1,n+1}\ x^{\alpha_{n+1}-1}) = \alpha_1 \ldots \alpha_{n+1} \sum_{j=1}^{n+1} c_{j,n+1} x^{\alpha_j-1},$$

which completes a proof of (31). The expression (32) for c.d.f. is immediate consequence of (31). This completes a proof of Lemma 2.

**2.4.** Proof of Theorem 1.

In view of Lemma 1, to prove formula (2) it is necessary and sufficient to show the equality

$$\log\big( (\mathcal{J}^{\{\beta_1,\ldots,\beta_n\}}(\nu))\widehat{\ }\big)(y) = \mathbb{E}[\log \hat{\nu}(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n} y)]$$
$$= \int_0^1 \log \hat{\nu}(ty) dr_{\{\beta_1,\ldots,\beta_n\}}(t), \quad y \in E'. \tag{37}$$

Of course, (37) holds for $n = 1$. Assume it is true for $n-1$. Then from Lemma 1 and the definition of the mapping $\mathcal{J}^{\{\beta_1,\ldots,\beta_n\}}$ (given before Theorem 1) we get

$$\log(\mathcal{J}^{\{\beta_1,\ldots,\beta_{n-1}\beta_n\}}(\nu))\widehat{\ }(y) = \int_0^1 \log(\mathcal{J}^{\{\beta_1,\ldots,\beta_{n1}\}}(\nu))\widehat{\ }(s\,y) ds^{\beta_n} =$$
$$\int_0^1 \mathbb{E}[\log \hat{\nu}(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_{n-1}^{1/\beta_{n-1}} sy)] ds^{\beta_n} = \mathbb{E}[\log \hat{\nu}(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n} y)]$$
$$= \int_0^1 \log \hat{\nu}(sy) dr_{\{\beta_1,\ldots,\beta_{n-1},\beta_n\}}(t), \tag{38}$$

which completes proof of (37) and consequently the formula (4). Explicit expressions for time changes $r_{\{\beta_1,\ldots,\beta_{n-1},\beta_n\}}(t)$ are given in Lemma 2, part (c).

**2.5.** Proof of Theorem 2.

First, putting $\Phi(y) = \log \hat{\nu}(y)$ ( the Lévy exponents of $\nu$) into (37) we get

$$i < y, a^{\{\beta_1,\ldots,\beta_n\}} > -\frac{1}{2} < y, R^{\{\beta_1,\ldots,\beta_n\}} y >$$
$$+ \int_{\mathbb{R}^d \setminus \{0\}} [e^{i<y,x>} - 1 - i < y, x > 1_B(x)] M^{\{\beta_1,\ldots,\beta_n\}}(dx)$$
$$= \mathbb{E}[\Phi(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n} y)]$$

Since for $0 < s \le 1$, we have that $1_B(sx) - 1_B(x) = 1_{\{1<||x||\le s^{-1}\}}(x)$ therefore from the above and (23) we get

$$a^{\{\beta_1,\ldots,\beta_n\}} = \mathbb{E}[U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n} (a + \int_{1<||x||\le (U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n})^{-1}} x M(dx))]$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j}\, a + \int_0^1 \int_{1<||x||\le s^{-1}} sx\, M(dx) dr_{\{\beta_1,\ldots,\beta_n\}}(s) \quad \text{(by (4))}$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j}\, a + \beta_1\beta_2 \ldots \beta_n \sum_{j=1}^{n} c_{j,n} \int_0^1 \int_{1<||x||\le s^{-1}} sx\, M(dx) s^{\beta_j - 1} ds \quad \text{(by (8))}$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j}\, a + \sum_{j=1}^{n} \frac{\beta_j}{\beta_j + 1} ( \prod_{k\ne j, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_j}) \, b_{M,\beta_j} \quad \text{(by (7) with z = -1)}$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{1+\beta_j} [\sum_{j=1}^{n} (a + b_{M,\beta_j}) \prod_{k\ne j, k=1}^{n} \frac{\beta_k + 1}{\beta_k - \beta_j}] = \sum_{j=1}^{n} C_{j,n}\, a^{\{\beta_j\}},$$

which proves the formula for the shift vector.

Similarly, for the Gaussian part, using again the identity (7) (with $z_j = \beta_j$ and $z = -2$) we get

$$R^{\{\beta_1,\ldots,\beta_n\}} = \mathbb{E}[(U_1^{1/\beta_1} \cdot U_2^{1/\beta_2} \cdot \ldots \cdot U_n^{1/\beta_n})^2]\, R = \prod_{j=1}^{n} \frac{\beta_j}{\beta_j + 2}\, R$$
$$= \prod_{j=1}^{n} \frac{\beta_j}{\beta_j + 2} (\sum_{l=1}^{n} \prod_{k\ne l, k=1}^{n} \frac{\beta_k + 2}{\beta_k - \beta_l})\, R = \sum_{l=1}^{n} ( \prod_{k\ne l, k=1}^{n} \frac{\beta_k}{\beta_k - \beta_l})\, R^{\{\beta_l\}} = \sum_{l=1}^{n} C_{l,n}\, R^{\{\beta_l\}},$$

which gives the formula for Gaussian covariance.

Finally for the Lévy spectral measure using (4) and (9) we have

$$
\begin{aligned}
&M^{\{\beta_1,\dots,\beta_n\}}(A) = \mathbb{E}[M((U_1^{1/\beta_1}\cdot U_2^{1/\beta_2}\cdot \ldots \cdot U_n^{1/\beta_n})^{-1}A)] \\
&= \int_0^1 T_s M(A) dr_{\{\beta_1,\dots,\beta_n\}}(s) = \beta_1\beta_2 \ldots \beta_n \sum_{j=1}^n c_{j,n} \int_0^1 M(s^{-1}A)\, s^{\beta_j -1} ds \\
&\qquad = \sum_{j=1}^n (\prod_{k\neq j,k=1}^n \frac{\beta_k}{\beta_k-\beta_j}) \int_0^1 M(s^{-1/\beta_j}A)\, ds = \sum_{j=1}^n C_{j,n}\, M^{\{\beta_j\}}(A),
\end{aligned}
$$

which completes the proof of Theorem 2.

**2.6.** Products of uniform distributions; part II.
For the proofs below note that for $a \neq 0$ we have the identity

$$
\int_s^1 w^{a-1}\,(-\log w)^{m-1} dw = \frac{(m-1)!}{a^m}\,\big[\,1 - s^a \sum_{k=0}^{m-1} \frac{(-a\log s)^k}{k!}\,\big], \;\; s>0, \quad (39)
$$

because both sides vanish at $s=1$ and have the same first derivative.

**Remark 6.** Note that for $a>0$, the expression on the right hand side of (39) in the square bracket, coincides on $(0,1]$ with the c.d.f. $F_{\{a,\dots,a\}}$ from (29); recall also Remark 2. Below we will use the same notation $F_{\{a,\dots,a\}}$ for $a<0$ but will not treat them as the c.d.f. of products of negative powers of independent uniform r.v's. Nevertheless first derivative of $F_{\{a,\dots,a\}}$ is still $f_{\{a,\dots,a\}}$ given by (28).

**Lemma 3.** *(i) For distinct positive reals $\beta_1,\beta_2,\dots,\beta_n,\alpha$ let us define*

$$
\begin{aligned}
&d^{(l)}_{j,n} \equiv d^{(l)}_{j,n}(\beta_1,\dots,\beta_n\,;\underbrace{\alpha,\dots,\alpha}_{l-times}) = \\
&\prod_{k\neq j\,,k=1}^{n} \frac{\beta_k}{\beta_k-\beta_j}\,(\frac{\alpha}{\alpha-\beta_j})^l = C_{j,n}(\beta_1,\dots,\beta_n)(\frac{\alpha}{\alpha-\beta_j})^l, \quad l=0,1,2,\dots. \quad (40)
\end{aligned}
$$

*Then for $n\geq 1$ and $m=1,2\ldots$ we get*

$$
f_{\{\beta_1,\dots,\beta_n,\underbrace{\alpha,\dots,\alpha}_{m-times}\}}(x) = \sum_{j=1}^n d^{(m)}_{j,n}\, f_{\{\beta_j\}}(x) - \alpha^{-1}\sum_{k=0}^{m-1}\big(\sum_{j=1}^n \beta_j\, d^{(m-k)}_{j,n}\big) f_{\{\underbrace{\alpha,\dots,\alpha}_{k+1-times}\}}(x), \quad (41)
$$

*and hence*

$$F_{\{\beta_1,\ldots,\beta_n,\underbrace{\alpha,\ldots,\alpha}_{m-times}\}}(s)=\sum_{j=1}^{n} d_{j,n}^{(m)}\,F_{\{\beta_j\}}(s)-\alpha^{-1}\sum_{k=0}^{m-1}\big(\sum_{j=1}^{n}\beta_j\,d_{j,n}^{(m-k)}\big)F_{\{\underbrace{\alpha,\ldots,\alpha}_{k+1-times}\}}(s)$$

$$=\sum_{j=1}^{n} d_{j,n}^{(m)}\,F_{\{\beta_j\}}(x)+F_{\{\underbrace{\alpha,\alpha,\ldots,\alpha}_{m-times}\}}(s)-\big[\sum_{j=1}^{n} d_{j,n}^{(m)}\,F_{\{\underbrace{\alpha-\beta_j,\ldots,\alpha-\beta_j}_{m-times}\}}\,F_{\{\alpha\}}(s)\,.$$

*(ii) For $k,l\geq 1$ and positive $\gamma\neq\alpha$ we have*

$$f_{\{\underbrace{\alpha,\ldots,\alpha}_{k-times},\underbrace{\gamma,\ldots,\gamma}_{l-times}\}}(z)=\Big(\frac{\alpha}{\alpha-\gamma}\Big)^k\Big(\frac{\gamma}{\gamma-\alpha}\Big)^l$$

$$\Big[\sum_{s=1}^{k}\frac{(l)_{k-s}}{(k-s)!}\Big(\frac{\alpha-\gamma}{\alpha}\Big)^s f_{\{\underbrace{\alpha,\ldots,\alpha}_{s-times}\}}(z)-\sum_{r=1}^{k+l-1} e_{r,\,k+l}\Big(\frac{\gamma-\alpha}{\gamma}\Big)^r f_{\{\underbrace{\gamma,\ldots,\gamma}_{r-times}\}}(z)\Big]$$

*where coefficients $e_{r,k+l}$ are given by*

$$e_{r,\,k+l}=\sum_{s=1}^{r\wedge k}(-1)^s\frac{(s)_{r-s}}{(r-s)!}\frac{(l)_{k-s}}{(k-s)!}=\sum_{s=1}^{r\wedge k}(-1)^s\binom{r-1}{s-1}\binom{l+k-s-1}{l-1},$$

*for $1\leq r\leq k+l-1$ and for $w\in\mathbb{R}$ and $m=1,2,\ldots,$ $(w)_m:=w(w+1)\ldots(w+m-1)$ denotes the Pochhammer symbol.*

*Proof.* Using (33), (31), (39), (40) and then (28) we get

$$f_{\{\beta_1,\ldots,\beta_n,\underbrace{\alpha,\ldots,\alpha}_{m-times}\}}(x)=\int_x^1 f_{\{\beta_1,\ldots,\beta_n\}}\big(\frac{x}{t}\big)t^{-1}f_{\{\alpha,\ldots,\alpha\}}(t)dt$$

$$=\frac{\alpha^m}{(m-1)!}\sum_{j=1}^{n}C_{j,n}\,\beta_j\,x^{\beta_j-1}\int_x^1 t^{\alpha-\beta_j-1}(-\log t)^{m-1}dt$$

$$=\sum_{j=1}^{n} d_{j,n}^{(m)}\,f_{\{\beta_j\}}-x^{\alpha-1}\sum_{j=1}^{n}\frac{C_{j,n}\,\beta_j}{(\alpha-\beta_j)^m}\sum_{k=0}^{m-1}\frac{((\beta_j-\alpha)\log x)^k}{k!})$$

$$=\sum_{j=1}^{n} d_{j,n}^{(m)}\,f_{\{\beta_j\}}-x^{\alpha-1}\sum_{k=0}^{m-1}\frac{(-\log x)^k}{k!}\sum_{j=1}^{n}C_{j,n}\,\beta_j\,\frac{\alpha^m}{(\alpha-\beta_j)^{m-k}}$$

$$=\sum_{j=1}^{n} d_{j,n}^{(m)}\,f_{\{\beta_j\}}(x)-\alpha^{-1}\sum_{k=0}^{m-1}\underbrace{\alpha^{k+1}x^{\alpha-1}\frac{(-\log x)^k}{k!}}\sum_{j=1}^{n}\beta_j\,d_{j,n}^{(m-k)}$$

$$=\sum_{j=1}^{n} d_{j,n}^{(m)}\,f_{\{\beta_j\}}-\alpha^{-1}\sum_{k=0}^{m-1}\big(\sum_{j=1}^{n}\beta_j\,d_{j,n}^{(m-k)}\big)\,f_{\{\underbrace{\alpha,\alpha,\ldots,\alpha}_{k+1-times}\}}(x)$$

where the probability density functions $f_{\{\beta_j\}}$ and $f_{\{\alpha,\alpha,...,\alpha\}}$ are given by (31) and (28) respectively. Thus we proved the formula (41).

From the above immediately follows the c.d.f. $F_{\{\beta_1,\beta_2,...,\beta_n,\alpha,...,\alpha\}}$ in the first form in Lemma 3 (i). For the second expression note that

$$- \alpha^{-1} \sum_{k=0}^{m-1} (\sum_{j=1}^{n} \beta_j \, d_{j,n}^{(m-k)} \, F_{\{\underbrace{\alpha, \alpha, ..., \alpha}_{k+1-times}\}}(s) =$$

$$- \alpha^{-1} \sum_{j=1}^{n} C_{j,n} \, \beta_j \sum_{k=0}^{m-1} (\frac{\alpha}{\alpha - \beta_j})^{m-k} \sum_{l=0}^{k} \frac{(-\alpha \log s)^l}{l!} \; s^{\alpha}$$

$$= -\alpha^{-1} \sum_{j=1}^{n} C_{j,n} \, \beta_j \sum_{l=0}^{m-1} \frac{(-\alpha \log s)^l}{l!} \sum_{k=l}^{m-1} (\frac{\alpha}{\alpha - \beta_j})^{m-k} \; s^{\alpha}$$

$$= - \sum_{j=1}^{n} C_{j,n} \sum_{l=0}^{m-1} \frac{(-\alpha \log s)^l}{l!} \, ((\frac{\alpha}{\alpha - \beta_j})^{m-l} - 1) \; s^{\alpha} = s^{\alpha} \sum_{l=0}^{m-1} \frac{(-\alpha \log s)^l}{l!}$$

$$- \sum_{j=1}^{n} C_{j,n} (\frac{\alpha}{\alpha - \beta_j})^{m} \sum_{l=0}^{m-1} \frac{(-(\alpha - \beta_j) \log s)^l}{l!} \, F_{\{\alpha\}}$$

$$= F_{\{\underbrace{\alpha, \alpha, ..., \alpha}_{m-times}\}}(s) - \sum_{j=1}^{n} C_{j,n} (\frac{\alpha}{\alpha - \beta_j})^{m} \, F_{\{\beta_j\}}(s) \, F_{\{\underbrace{\alpha - \beta_j, ..., \alpha - \beta_j}_{m-times}\}}(s),$$

which completes the proof of the part (i).

Before the proof of part (ii), for an ease of notation, let us recall here the Pochhammer symbol

$$\text{for } a \in \mathbb{R} \text{ and } n = 1, 2, ... \;\; (a)_n := a(a+1)(a+2)...(a+n-1), \;\; (a)_0 := 1. \quad (42)$$

For $k \geq 1$, $l \geq 1$ and $\gamma \neq \alpha$ and $\gamma, \alpha > 0$ using (33) and (39)we have

$$f_{\{\underbrace{\alpha, ..., \alpha}_{k-times}, \underbrace{\gamma, ..., \gamma}_{l-times}\}}(z) = \int_z^1 \; f_{\{\underbrace{\alpha, ..., \alpha}_{k-times}\}}(\frac{z}{x}) \frac{1}{x} \, f_{\{\underbrace{\gamma, ..., \gamma}_{l-times}\}}(x) dx$$

$$= \frac{\alpha^k \gamma^l}{(k-1)!(l-1)!} \, z^{\alpha-1} \int_z^1 x^{\gamma-\alpha-1} (-\log z + \log x)^{k-1} (-\log x)^{l-1} \, dx$$

$$= \frac{\alpha^k \gamma^l}{(k-1)!(l-1)!} \, z^{\alpha-1} \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s} (-\log z)^{k-1-s} \int_z^1 x^{\gamma-\alpha-1} \, (-\log x)^{s+l-1} dx$$

$$= \frac{\alpha^k \gamma^l}{(k-1)!(l-1)!} z^{\alpha-1} \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s} (-\log z)^{k-1-s} \frac{(s+l-1)!}{(\gamma-\alpha)^{s+l}}$$

$$\cdot \left[1 - z^{\gamma-\alpha} \sum_{i=0}^{s+l-1} \frac{(-(\gamma-\alpha)\log z)^i}{i!}\right]$$

$$= \alpha^k (\frac{\gamma}{\gamma-\alpha})^l \sum_{s=0}^{k-1} \frac{(l)_s}{s!} \frac{1}{(\alpha-\gamma)^s} \frac{(-\log z)^{k-1-s}}{(k-1-s)!} z^{\alpha-1}[1-F_{\{\underbrace{\gamma-\alpha, ..., \gamma-\alpha}_{l+s-times}\}}(z)]$$

$$= (\frac{\gamma}{\gamma-\alpha})^l \sum_{s=0}^{k-1} \frac{(l)_s}{s!} (\frac{\alpha}{\alpha-\gamma})^s f_{\{\underbrace{\alpha, ..., \alpha}_{k-s-times}\}}(z)[1-F_{\{\underbrace{\gamma-\alpha, ..., \gamma-\alpha}_{l+s-times}\}}(z)]$$

$$= (\frac{\alpha}{\alpha-\gamma})^k (\frac{\gamma}{\gamma-\alpha})^l \sum_{s=1}^{k} \frac{(l)_{k-s}}{(k-s)!} (\frac{\alpha-\gamma}{\alpha})^s f_{\{\underbrace{\alpha, ..., \alpha}_{s-times}\}}(z)[1-F_{\{\underbrace{\gamma-\alpha, ..., \gamma-\alpha}_{l+k-s-times}\}}(z)] \tag{43}$$

For further simplification of (43) note that the function (28) and (29) are well defined for $a, b \in \mathbb{R} \setminus \{0\}$. ( See also Remark 6.) Hence, if $a+b \neq 0$ and $p, q \geq 1$ then

$$f_{\{\underbrace{a, ..., a}_{p-times}\}}(z) F_{\{\underbrace{b, ..., b}_{q-times}\}}(z) = (\frac{a}{a+b})^p \sum_{j=0}^{q-1} \frac{(p)_j}{j!} (\frac{b}{a+b})^j f_{\{\underbrace{a+b ... a+b}_{p+j-times}\}}(z)$$

$$= (\frac{a}{b})^p \sum_{j=p}^{p+q-1} \frac{(p)_{j-p}}{(j-p)!} (\frac{b}{a+b})^j f_{\{\underbrace{a+b ... a+b}_{j-times}\}}(z) \tag{44}$$

Using the above with $p = s$, $q = l + k - s$, $a = \alpha$ and $b = \gamma - \alpha$ we get

$$f_{\{\underbrace{\alpha, ..., \alpha}_{s-times}\}}(z)[1-F_{\{\underbrace{\gamma-\alpha, ..., \gamma-\alpha}_{l+k-s-times}\}}(z)]$$

$$= f_{\{\underbrace{\alpha, ..., \alpha}_{s-times}\}}(z) - (\frac{\alpha}{\gamma-\alpha})^s \sum_{r=s}^{l+k-1} \frac{(s)_{r-s}}{(r-s)!} (\frac{\gamma-\alpha}{\gamma})^r f_{\{\underbrace{\gamma, ..., \gamma}_{r-times}\}}(z).$$

Hence and from (43) we get

$$f_{\{\underbrace{\alpha, ..., \alpha}_{k-times}, \underbrace{\gamma, ..., \gamma}_{l-times}\}}(z) = (\frac{\alpha}{\alpha-\gamma})^k (\frac{\gamma}{\gamma-\alpha})^l \sum_{s=1}^{k} \frac{(l)_{k-s}}{(k-s)!} (\frac{\alpha-\gamma}{\alpha})^s$$

$$\cdot [f_{\{\underbrace{\alpha, ..., \alpha}_{s-times}\}}(z) - (\frac{\alpha}{\gamma-\alpha})^s \sum_{r=s}^{l+k-1} \frac{(s)_{r-s}}{(r-s)!} (\frac{\gamma-\alpha}{\gamma})^r f_{\{\underbrace{\gamma, ..., \gamma}_{r-times}\}}(z)]. \tag{45}$$

Changing the order of summation in the second summand in (45) we get

$$\sum_{s=1}^{k} \frac{(l)_{k-s}}{(k-s)!} (\frac{\alpha-\gamma}{\alpha})^s (\frac{\alpha}{\gamma-\alpha})^s \sum_{r=s}^{l+k-1} \frac{(s)_{r-s}}{(r-s)!} (\frac{\gamma-\alpha}{\gamma})^r f_{\{\underbrace{\gamma,...,\gamma}_{r-times}\}}(z)$$

$$= \sum_{r=1}^{k} [\sum_{s=1}^{r} (-1)^s \frac{(s)_{r-s}}{(r-s)!} \frac{(l)_{k-s}}{(k-s)!}] (\frac{\gamma-\alpha}{\gamma})^r f_{\{\underbrace{\gamma,...,\gamma}_{r-times}\}}(z)$$

$$+ \sum_{r=k+1}^{l+k-1} [\sum_{s=1}^{k} (-1)^s \frac{(s)_{r-s}}{(r-s)!} \frac{(l)_{k-s}}{(k-s)!}] (\frac{\gamma-\alpha}{\gamma})^r f_{\{\underbrace{\gamma,...,\gamma}_{r-times}\}}(z).$$

Defining the coefficients

$$e_{r,k+l} = \begin{cases} \sum_{s=1}^{r} (-1)^s \frac{(s)_{r-s}}{(r-s)!} \frac{(l)_{k-s}}{(k-s)!} & \text{for } 1 \le r \le k \\ \sum_{s=1}^{k} (-1)^s \frac{(s)_{r-s}}{(r-s)!} \frac{(l)_{k-s}}{(k-s)!} & \text{for } k+1 \le r \le k+l-1, \end{cases}$$

finally give us that

$$f_{\{\underbrace{\alpha,...,\alpha}_{k-times},\underbrace{\gamma,...,\gamma}_{l-times}\}}(z) = (\frac{\alpha}{\alpha-\gamma})^k (\frac{\gamma}{\gamma-\alpha})^l$$

$$[\sum_{s=1}^{k} \frac{(l)_{k-s}}{(k-s)!} (\frac{\alpha-\gamma}{\alpha})^s f_{\{\underbrace{\alpha,...,\alpha}_{s-times}\}}(z) - \sum_{r=1}^{k+l-1} e_{r,k+l} (\frac{\gamma-\alpha}{\gamma})^r f_{\{\underbrace{\gamma,...,\gamma}_{r-times}\}}(z)],$$

which completes a proof of (ii) in Lemma 3.

**2.7.** A note on a more general case.

For distinct $\beta_1, ..., \beta_n, \alpha, \gamma$ using again (33) and Lemma 3 we obtain

$$f_{\{\beta_1,\beta_2,...,\beta_n,\underbrace{\alpha,...,\alpha}_{m-times},\underbrace{\gamma,...,\gamma}_{l-times}\}}(z) = \int_z^1 f_{\{\beta_1,...,\beta_n,\underbrace{\alpha,...,\alpha}_{m-times}\}}(\frac{z}{x}) \frac{1}{x} f_{\{\underbrace{\gamma,...,\gamma}_{l-times}\}}(x)dx$$

$$= \sum_{j=1}^{n} d_{j,n}^{(m)} f_{\{\beta_j,\underbrace{\gamma,...,\gamma}_{l-times}\}}(z) - \alpha^{-1} \sum_{k=0}^{m-1} (\sum_{j=1}^{n} \beta_j d_{j,n}^{(m-k)}) f_{\{\underbrace{\alpha,...,\alpha}_{k+1-times},\underbrace{\gamma,...,\gamma}_{l-times}\}}(z), \tag{46}$$

where p.d.f. $f_{\{\beta_j,\underbrace{\gamma,...,\gamma}_{l-times}\}}(z)$ and $f_{\{\underbrace{\alpha,...,\alpha}_{k+1-times},\underbrace{\gamma,...,\gamma}_{l-times}\}}(z)$ are given in Lemma 3. Thus, in principle, for an arbitrary finite set $\mathbb{A}$ of positive reals we have a formula for $f_{\mathbb{A}}(z)$. However, our expressions become quite complicated; comp. Remark 3.

**3. Concluding remarks.**

(a) The formulae for the time scale change $r$ in Theorem 1, Proposition 2 and in the section 2.7 are quite involved and not very transparent. Maybe using the notation $\rho_{\mathbb{B}}$ given in Remark 3, formula (17), would clarify the expositions.

(b) The ideas of the proof of Theorem 1 seems to be applicable to compositions of mappings $I^{h,r}_{(a,b]}$, where the time scale change $r$ is a c.d.f. Namely, from Lemma 1, if $r_1$ and $r_2$ are c.d.f. of rv's $X_1$, $X_2$ concentrated on intervals $(a_1, b_1]$ and $(a_2, b_2]$, respectively, then

$$\log \big(I^{h_1,r_1}_{(a_1,b_1]}(I^{h_2,r_2}_{(a_2,b_2]}(\nu))\widehat{\big)}(y) = \mathbb{E}[\log \hat{\nu}(h_1(X_1)h_2(X_2)y)] \\ = \int_{(a_3,b_3]} \log \hat{\nu}(sy)dr_3(s) = \log \big(I^{t,\,r_3}_{(a_3,b_3]}(\nu)\widehat{\big)}, \quad (47)$$

where $(a_3, b_3] = r_1((a_1, b_1]) \cdot r_2((a_2, b_2])$ and $r_3$ is the cumulative distribution function of rv $h_1(X_1)h_2(X_2)$. However, if $h_1(x_1) \cdot h_2(x_2) = h_3(g(x_1, x_2))$ for some measurable functions $g : \mathbb{R}^2 \to \mathbb{R}$ and $h_3 : \mathbb{R}^+ \to \mathbb{R}$ then

$$I^{h_1,r_1}_{(a_1,b_1]} \circ I^{h_2,r_2}_{(a_2,b_2]} = I^{h_3,\tilde{r}_4}_{(a_3,b_3]}(\nu),$$

where $\tilde{r}_4$ is the c.d.f. of rv $g(X_1, X_2)$ and $\circ$ denotes the composition of the random integral mappings (provided the mappings in questions are well defined, i.e., appropriate domains and ranges coincide).

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Institute of Mathematics, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: zjjurek@math.uni.wroc.pl www.math.uni.wroc.pl/∼zjjurek